

Explicit Weak-Type Constants via Partial Balayage

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Overview

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3. Applications to singular integrals and maximal functions.

Both principles use the same partial balayage decomposition.

Weak-Type Constants: Linear Operators

All entries are upper bounds for $\|T\|_{1 \rightarrow 1, \infty}$.

Operator	Bound
Vector Riesz transform	2
Beurling–Ahlfors transform	2
Second-order Riesz transform, $n = 2$	$3\sqrt{6}/4$
Second-order Riesz transform, $n \geq 2$	$C_n^{(2)} < 2$
Traceless second-order Riesz transform	$2\sqrt{1 - 1/n}$
Leray and gradient projections	1.805359...

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$$C_n^{(2)} = \frac{1}{a_n} + \frac{(n-1)a_n}{n - a_n^2}, \quad a_n^2 = \frac{2n}{n+1 + \sqrt{(n+1)^2 + 4(n-2)}}.$$

Two partial balayage principles. Only the Riesz and Beurling applications will be discussed.

Weak-Type Constants: Geometric Maximal Operators

Averaging sets	Bound
Intervals, $n = 1$	2
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Euclidean balls, $n \geq 3$	$(n/2)^{n/(n-2)}$
Axis-parallel squares, $n = 2$	3.879

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These are explicit bounds. Optimality of a comparison kernel does not imply the sharp operator constant.

Two partial balayage principles; Lean formalization; Palomar: two principles.

Weak-Type Constants: Semigroup Maximal Operators

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There are integral formulas for the bounds in every dimension. For comparison, maximal *time averages* of the heat semigroup have weak-type constant 1 by HDS.

Two partial balayage principles, Sections 4.4–4.5. General formulas appear in the backup slides.

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Article Lean formalization

Palomar: two principles Palomar: squares

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$$D = \{u > 0\}, \quad \sigma = \kappa.$$

Conservation of mass gives the key estimate:

$$\boxed{\kappa|D| = \int_D \sigma \leq \int \sigma = \int f.}$$

For the Laplacian, $\mathcal{A} = \Delta$. Fractional and coordinate-stable generators also occur below.

The Decomposition Lemma

Theorem (Partial balayage, including Hilbert-valued data)

Let E be finite-dimensional, $0 < \alpha \leq 2$, and $f \in L^1(\mathbb{R}^n; E) \cap L^2(\mathbb{R}^n; E)$. For every $\kappa > 0$,

$$f = \mu + (-\Delta)^{\alpha/2} u, \quad u \in H^\alpha \cap L^1,$$

with

$$\|\mu\|_\infty \leq \kappa, \quad \|\mu\|_1 \leq \|f\|_1,$$

and, on $\Omega = \{|u|_E > 0\}$,

$$\mu = \kappa \frac{u}{|u|_E}, \quad \boxed{\kappa |\Omega| \leq \|f\|_1}.$$

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For scalar $f \geq 0$, one has $u, \mu \geq 0$ and $\int \mu = \int f$.

Two partial balayage principles, Theorem 1.1, extending Ouyang–Spector–Stockdale (2026).

Where the Decomposition Comes From

Minimize, over $H^{\alpha/2} \cap L^1$,

$$J(v) = \frac{1}{2} \left\| (-\Delta)^{\alpha/4} v \right\|_2^2 + \kappa \int |v|_E - \operatorname{Re} \int \langle f, v \rangle_E.$$

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Test against expanding cutoffs to obtain $\|\mu\|_1 \leq \|f\|_1$. Saturation then gives $\kappa|\Omega| \leq \|f\|_1$.

Existence uses the variational compactness argument. For $\alpha = 2$, the energy is $\frac{1}{2} \int |\nabla v|^2$.

The Sobolev Zero-Set Lemma

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If $v \in W_{\text{loc}}^{1,p}$, $1 \leq p < \infty$, then

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If $v \in W_{\text{loc}}^{2,p}$, apply this twice:

$$\nabla v = 0 \text{ on } \{v = 0\}, \quad D^2 v = 0 \text{ on } \{v = 0\}.$$

Principle 1: Linear Operators

Theorem

Suppose $T : L^2(\mathbb{R}^n; E) \rightarrow L^2(\mathbb{R}^n; F)$ is linear and

$$\|T\|_{2 \rightarrow 2} \leq M.$$

Assume that for some $0 < \alpha \leq 2$,

$$T(-\Delta)^{\alpha/2} u = 0 \quad \text{a.e. on } \{u = 0\}, \quad u \in H^\alpha.$$

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$$\boxed{\|T\|_{1 \rightarrow 1, \infty} \leq 2M.}$$

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A differential operator of the appropriate order satisfies the required vanishing by the zero-set lemma.

Proof of the Linear Principle

Decompose $f = \mu + (-\Delta)^{\alpha/2} u$ at height κ . Outside $\Omega = \{|u| > 0\}$, the assumed vanishing gives

$$Tf = T\mu.$$

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Thus

$$\begin{aligned} \lambda |\{|Tf| > \lambda\}| &\leq \lambda |\Omega| + \lambda^{-1} \|T\mu\|_2^2 \\ &\leq \left(\frac{\lambda}{\kappa} + \frac{M^2 \kappa}{\lambda} \right) \|f\|_1. \end{aligned}$$

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Here $\|\mu\|_2^2 \leq \kappa \|\mu\|_1 \leq \kappa \|f\|_1$.

$$\kappa = \lambda/M \implies \lambda |\{|Tf| > \lambda\}| \leq 2M \|f\|_1.$$

For $M > 0$. Density extends the estimate from $L^1 \cap L^2$ to L^1 .

Application: The Vector Riesz Transform

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This is not a new result of mine. It reinterprets the theorem of Ouyang, Spector, and Stockdale through Principle 1.

Ouyang–Spector–Stockdale, arXiv:2608.18068 (2026), Theorem 1.1.

Application: The Beurling–Ahlfors Transform

On $\mathbb{C}$,

$$Bf(z) = -\frac{1}{\pi} \text{p. v.} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dA(w).$$

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For real-valued inputs, Guerra proved the sharp constant 2 by a different method in August.

Guerra, arXiv:2608.23388 (2026).

Markov Convolution Semigroups

A convolution semigroup is a family of averaging operators

$$P_t g(x) = \int_{\mathbb{R}^n} g(x - y) d\nu_t(y), \quad t > 0,$$

with $P_s P_t = P_{s+t}$ and $P_t g \rightarrow g$ in L^2 as $t \downarrow 0$.

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► **Markov and conservative:** ν_t is a probability measure.

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► **Symmetric:** $\nu_t(E) = \nu_t(-E)$ for every Borel set E .

$$\langle P_t g, h \rangle_{L^2} = \langle g, P_t h \rangle_{L^2}.$$

The mass identity applies to integrable data. Convolution means the averaging rule is invariant under translations.

The Generator

The generator is the instantaneous rate of change:

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Its domain consists of the L^2 functions for which this limit exists in L^2 .

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Other examples are

$$\mathcal{A} = -(-\Delta)^{\alpha/2}, \quad \mathcal{A} = -\sum_j |D_j|^\alpha, \quad 0 < \alpha < 2.$$

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Our sign convention is $L = -\mathcal{A}$:

$$\partial_t(P_t g) = \mathcal{A}P_t g, \quad P_t = e^{t\mathcal{A}} = e^{-tL}.$$

Homogeneity and Dilations

Definition

The operator $\mathcal{A}$ is homogeneous of order $\alpha > 0$ if

$$\mathcal{A}[g(\cdot/r)](x) = r^{-\alpha}(\mathcal{A}g)(x/r), \quad r > 0.$$

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Thus the condition for one kernel gives the condition at every scale:

$$\mathcal{A}K \geq 0 \text{ off } 0 \implies \mathcal{A}K_r \geq 0 \text{ off } 0 \text{ for all } r > 0.$$

This is why the same comparison argument applies to every dilation in the maximal operator.

The Setting for Principle 2

Let $\mathcal{A}$ be a homogeneous generator as above. Assume partial balayage is available: for smooth compactly supported $f \geq 0$,

$$\begin{aligned} f &= \sigma - \mathcal{A}u, \quad u \geq 0, \quad 0 \leq \sigma \leq \kappa, \\ \sigma &= \kappa \text{ on } D = \{u > 0\}, \quad \kappa|D| \leq \|f\|_1. \end{aligned}$$

This decomposition holds for the examples used in this talk.

Principle 2: Maximal Operators

Theorem

Let $K \geq 0$ be even and integrable, and suppose

$$\mathcal{A}K \geq 0 \quad \text{in distributions on } \mathbb{R}^n \setminus \{0\}.$$

Define

$$K_r(x) = r^{-n}K(x/r), \quad M_K f = \sup_{r>0} K_r * |f|.$$

Then

$$\|M_K\|_{1 \rightarrow 1, \infty} \leq \int K.$$

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The sign condition permits a negative singularity at the origin.

Proof: The Required Inequality

Let $M = \int K > 0$. Use partial balayage at height $\kappa = \lambda/M$:

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Since $0 \leq \sigma \leq \kappa$, at each scale we have

$$K_r * f = K_r * \sigma - K_r * \mathcal{A}u \leq \lambda - K_r * \mathcal{A}u.$$

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Thus the estimate would follow from

$$\boxed{u \geq 0, \quad u = 0 \text{ on } D^c \implies K_r * \mathcal{A}u \geq 0 \text{ on } D^c.}$$

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Indeed, this would give $\{M_K f > \lambda\} \subseteq D$, and hence

$$\lambda|\{M_K f > \lambda\}| \leq \lambda|D| \leq M \|f\|_1.$$

The remaining task is precisely a minimum principle for $\mathcal{B}_r v = K_r * \mathcal{A}v$. All zero-set statements hold almost everywhere.

A Simple Minimum-Principle Example

Suppose the kernel satisfies

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Move the generator onto the kernel:

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This proves the required sign directly in this special case.

For nonsmooth u , these identities and inequalities hold almost everywhere. The general distributional hypothesis needs the argument that follows.

The Minimum Principle Behind the Estimate

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At a smooth minimum, $v(x) = 0$ and $\nabla v(x) = 0$, so v vanishes quadratically. A cutoff removes the restriction that v vanish *near* x .

The cutoff calculation is included in the backup slides. The rough potential u requires the domain argument below.

From the Minimum Principle to the Maximum Principle

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Choose $\chi_R = 1$ on its support and at x .

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Thus

$$\boxed{v(x) = \max v \geq 0 \implies \mathcal{B}v(x) \leq 0.}$$

This is the positive maximum principle required by Courrège's theorem.

Courrège's Representation Theorem

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$$\begin{aligned} \mathcal{B}v(x) = & -cv(x) + b \cdot \nabla v(x) + \operatorname{tr}(QD^2v(x)) \\ & + \int_{\mathbb{R}^n \setminus \{0\}} \left[v(x+y) - v(x) \right. \\ & \left. - \mathbf{1}_{\{|y|<1\}} y \cdot \nabla v(x) \right], d\nu(y). \end{aligned}$$

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Here $c \geq 0$, $b \in \mathbb{R}^n$, Q is positive semidefinite, and

$$\nu \geq 0, \quad \int \min(1, |y|^2) d\nu(y) < \infty.$$

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Here $c \geq 0$, $b \in \mathbb{R}^n$, Q is positive semidefinite, and

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These are, respectively, killing, drift, diffusion, and jump terms.

Courrège (1964). Evenness is not a hypothesis of this theorem. This is the Lévy–Khintchine generator form.

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since the jump measure has finite mass outside the unit ball. But $\|\mathcal{B}\chi_R\|_\infty \rightarrow 0$, as shown on the previous slides. Hence

$$\boxed{c = 0.}$$

The resulting convolution semigroup loses no mass: $Q_t 1 = 1$. Its kernels are probability measures.

Symmetry and the Minimum Principle

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- ▶ Both translated values in the jump term are nonnegative.

Thus $\mathcal{B}v(x) \geq 0$. The representation gives a positive symmetric semigroup $Q_t = e^{t\mathcal{B}}$, which also handles the nonsmooth potential.

The Auxiliary Generator and Its Domain

Write $\widehat{\mathcal{A}}v = -\psi\widehat{v}$. Since $\mathcal{B}v = K * \mathcal{A}v$,

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Let $Q_t = e^{t\mathcal{B}}$ denote the semigroup generated by $\mathcal{B}$. The definition of the generator then gives

$$\boxed{\frac{Q_t u - u}{t} \longrightarrow \mathcal{B}u \quad \text{in } L^2.}$$

The new semigroup Q_t is distinct from the original semigroup $P_t = e^{t\mathcal{A}}$.

The Minimum Principle on the Zero Set

The new semigroup averages against nonnegative probability measures:

$$Q_t u(x) = \int u(x - y) d\rho_t(y), \quad \rho_t \geq 0, \quad \rho_t(\mathbb{R}^n) = 1.$$

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The quotients converge in L^2 to $\mathcal{B}u$. Choose $t_j \downarrow 0$ along which convergence also holds almost everywhere. Hence

$$\mathcal{B}u \geq 0 \quad \text{a.e. on } D^c.$$

Hopf–Dunford–Schwartz: Heat-Semigroup Case

Let

$$h_s(x) = (4\pi s)^{-n/2} e^{-|x|^2/(4s)}, \quad A_t f = \frac{1}{t} \int_0^t h_s * f \, ds.$$

Theorem (Continuous-time HDS for the heat semigroup)

For $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$,

$$\lambda \left| \left\{ \sup_{t>0} A_t |f| > \lambda \right\} \right| \leq \|f\|_1.$$

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HDS holds for general Dunford–Schwartz semigroups. The proof here establishes its heat-semigroup instance.

HDS in One Kernel Calculation

The time-average kernel is

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Moreover $K_t(x) = t^{-n/2} K_1(x/\sqrt{t})$. Principle 2 with K_1 gives the constant 1. Equivalently, if $f = \sigma - \Delta u$, then on $\{u = 0\}$,

$$A_t f = A_t \sigma - \frac{h_t * u}{t} \leq \lambda.$$

A Kernel Optimization Problem

For an even nonnegative kernel φ , set

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Every admissible majorant $K \geq \varphi$ gives $M_\varphi \leq M_K$.

The Optimization Bound

$$\|M_\varphi\|_{1 \rightarrow 1, \infty} \leq C_{\mathcal{A}}(\varphi),$$

where

$$C_{\mathcal{A}}(\varphi) = \inf \left\{ \int K : \begin{array}{l} K \in L^1 \text{ is even, } K \geq \varphi, \\ \mathcal{A}K \geq 0 \text{ on } \mathbb{R}^n \setminus \{0\} \end{array} \right\}.$$

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We now solve this optimization problem for ball indicators and $\mathcal{A} = \Delta$.

Explicit Bounds for Centered Ball Averages

$$M_{B,n}f(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B(x,r)} |f(y)| \, dy, \quad \omega_n = |B(0,1)|.$$

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Theorem

$$\|M_{B,n}\|_{1 \rightarrow 1, \infty} \leq \begin{cases} 2, & n = 1, \\ e, & n = 2, \\ (n/2)^{n/(n-2)}, & n \geq 3. \end{cases}$$

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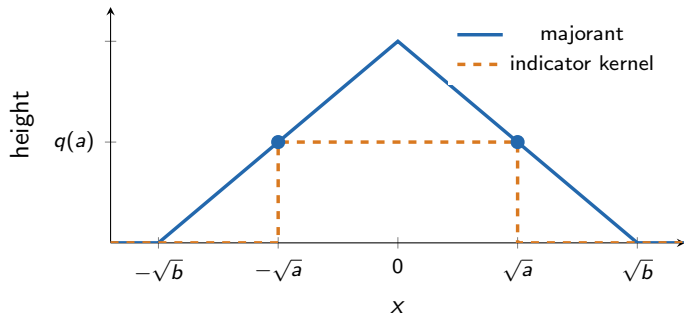
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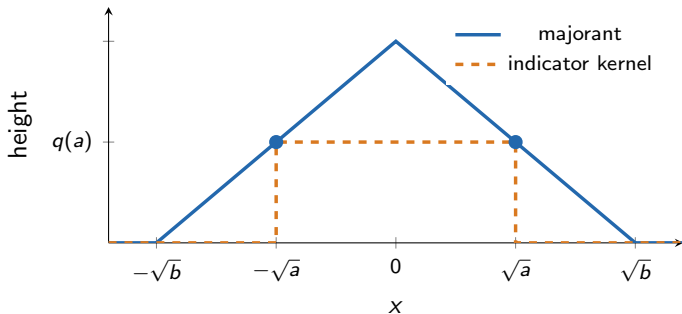
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These values equal $C_{\Delta}(\omega_n^{-1} \mathbf{1}_{B_1})$. In dimension one, the actual sharp operator constant is smaller.

The Interval Majorant



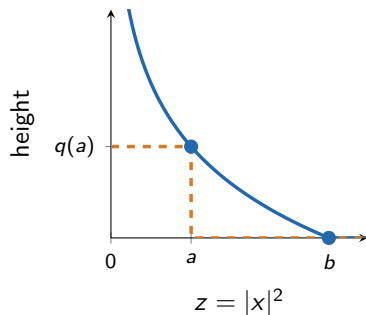
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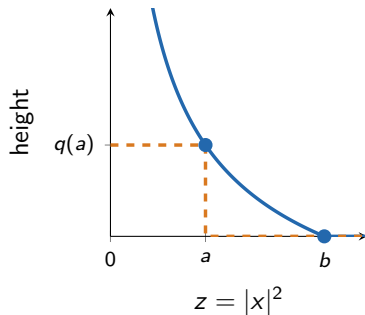
$$K(x) = (1 - |x|/2)_+, \quad a = 1, \quad b = 4a, \quad \int K = 2.$$

The Ball Majorants: Dimensions Two and Higher

$n = 2$: logarithmic



$n \geq 3$: Newtonian



Blue: majorant. Dashed orange: indicator. Both cases use

$$k_*(z) = q(a) \frac{s(b) - s(z)}{s(b) - s(a)} \quad (0 < z < b), \quad k_* = 0 \quad (z \geq b).$$

Symbolic axes: a, b are squared radii and $b = \rho_n a$. Heights are normalized by $q(a)$. The Newtonian panel illustrates $n = 3$.

The Explicit Kernels

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$$K_2(x) = \frac{2}{\pi} \left(\log \frac{\sqrt{e}}{|x|} \right)_+.$$

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$$K_n(x) = \frac{1}{\omega_n} \frac{(|x|^{2-n} - R_n^{2-n})_+}{1 - R_n^{2-n}}, \quad R_n^{2-n} = \frac{2}{n}.$$

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Each kernel majorizes $\omega_n^{-1} \mathbf{1}_{B_1}$, is harmonic inside its support away from zero, and has an upward derivative jump at the outer boundary.

$$\Delta K_n = \text{positive boundary measure} - c\delta_0, \quad c > 0.$$

Polar integration gives $\int K_2 = e$ and $\int K_n = R_n^n$ for $n \geq 3$.

When the Infimum Is Attained

Let $\varphi(x) = q(|x|^2)$, with q left-continuous at $a > 0$. Put

$$\rho_n = \begin{cases} (n/2)^{2/(n-2)}, & n \neq 2, \\ e, & n = 2, \end{cases} \quad b = \rho_n a.$$

Theorem

An admissible radial majorant K_* attains $C_\Delta(\varphi)$ if it

- 1. is harmonic for $0 < |x| < \sqrt{b}$,*
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We prove a lower bound for every competitor, then show equality for K_* .

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Average an arbitrary admissible K over orthogonal transformations:

$$\overline{K}(x) = \int_{O(n)} K(Rx) d\nu(R),$$

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- ▶ $\overline{K} \geq \varphi$, since φ is radial.
- ▶ $\int \overline{K} = \int K$, since rotations preserve volume.
- ▶ $\Delta \overline{K} \geq 0$ away from zero, since

$$\langle \Delta \overline{K}, \psi \rangle = \int_{O(n)} \langle \Delta K, \psi \circ R^{-1} \rangle d\nu(R) \geq 0$$

for nonnegative tests supported away from zero.

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for nonnegative tests supported away from zero.

We may therefore write each competitor as $K(x) = k(|x|^2)$.

Step 2: A Change of Radial Variable

Set

$$s(z) = \begin{cases} \frac{z^{1-n/2}}{1-n/2}, & n \neq 2, \\ \log z, & n = 2. \end{cases}$$

Since $s'(z) = z^{-n/2} > 0$, it is invertible onto its image.

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Since $s'(z) = z^{-n/2} > 0$, it is invertible onto its image. Define the transformed profile by **actual substitution**:

$$h(t) = k(s^{-1}(t)).$$

Step 2: A Change of Radial Variable

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The chain rule gives

$$h''(s(z)) = \frac{z^{n-1}}{4} \Delta K(x), \quad |x|^2 = z.$$

Thus subharmonicity makes h convex. Harmonicity makes h affine.

For example, in 2D: $s(z) = \log z$, so $h(t) = k(e^t)$. The statement also holds distributionally.

Step 3: Why the Ratio Is $b/a = \rho_n$

Let X be uniform in $B_{\sqrt{b}}$, and $Z = |X|^2$. Then

$$p_Z(z) = \frac{n}{2b^{n/2}} z^{n/2-1}, \quad 0 < z < b.$$

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Requiring $\mathbb{E}s(Z) = s(a)$ gives $b/a = (n/2)^{2/(n-2)}$. In 2D, Z is uniform on $(0, b)$, so

$$\mathbb{E} \log Z = \log b - 1 = \log a \quad \Longleftrightarrow \quad b = ea.$$

Step 4: Jensen Bounds Every Competitor

For $h = k \circ s^{-1}$, convexity gives

$$\begin{aligned}\frac{1}{|B_{\sqrt{b}}|} \int_{B_{\sqrt{b}}} K &= \mathbb{E}k(Z) = \mathbb{E}h(s(Z)) \\ &\geq h(\mathbb{E}s(Z)) = k(a) \geq q(a).\end{aligned}$$

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The last inequality follows from a.e. majorization, continuity of k , and left continuity of q at a . Outside the ball, simply use $K \geq \varphi$. Hence

$$\int K \geq |B_{\sqrt{b}}|q(a) + \int_{|x| \geq \sqrt{b}} \varphi(x) dx.$$

Step 5: The Proposed Kernel Attains the Bound

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Therefore

$$C_{\Delta}(\varphi) = \int K_* = |B_{\sqrt{b}}|q(a) + \int_{|x| \geq \sqrt{b}} \varphi.$$

For the normalized unit ball, $a = 1$, $q(a) = 1/\omega_n$, and the tail is zero. Thus

$$C_{\Delta}(\omega_n^{-1} \mathbf{1}_{B_1}) = b^{n/2} = \rho_n^{n/2}.$$

Comparison with Classical Bounds

Method	Bound for centered balls
Basic Vitali selection	3^n
Refined classical covering	2^n
Stein–Strömberg (1983)	$O(n)$
Laplacian majorant	$(n/2)^{n/(n-2)} \sim n/2$
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The $O(\cdot)$ entries describe dimensional growth; they are not numerical constants for a fixed dimension. In 1D, Melas proved the sharp centered constant

$$\frac{11 + \sqrt{61}}{12} = 1.567520\dots < 2.$$

Stein–Strömberg (1983); Spector–Stockdale, arXiv:2609.05377; Melas (2003); Tao, Notes 5, Exercise 42.

Why the HDS Comparison Cannot Improve This Constant

The classical heat-average comparison has the form

$$\omega_n^{-1} \mathbf{1}_{B_1} \leq CK_t, \quad K_t = \frac{1}{t} \int_0^t h_s \, ds.$$

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So any proof using this *single-kernel domination by heat time averages* cannot beat our optimal majorant. Spector–Stockdale use additional information from the heat residual, beyond this comparison.

This limitation concerns that comparison mechanism, not every possible use of HDS.

Squares: Change the Geometry and the Generator

Let $D_\diamond = \{(u, v) : |u| + |v| \leq 1\}$, of area 2. The map

$$(u, v) \longmapsto (u + v, u - v)$$

sends this diamond to $[-1, 1]^2$ and preserves the weak-type constant after changing variables.

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Principle 2 reduces the problem to

$$c_2 \leq \inf \left\{ \frac{1}{2} \int K : K \geq \mathbf{1}_{D_\diamond}, K \text{ even, } \mathcal{A}_\alpha K \geq 0 \text{ off } 0 \right\}.$$

Lin, $c_2 < 4$, Sections 1–2.

Squares: An Explicit Certified Kernel

Take $\alpha = 1$, and write $r = |u| + |v|$. Set

$$R = \frac{3797}{2000}, \quad \varepsilon = \frac{159411}{200000},$$

$$K(u, v) = \frac{(R/r - 1)_+}{R - 1} - \varepsilon(1 - r)_+^2.$$

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Diamond polar integration gives

$$c_2 \leq \frac{1}{2} \int K = \frac{R^2}{R - 1} - \frac{\varepsilon}{6} < 3.878624 < 3.879 < 4.$$

Lin, $c_2 < 4$, Section 3; Lean: squares; Palomar: squares.

Squares: What the Certificate Establishes

The explicit comparison kernel improves the classical covering bound:

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The verification includes majorization, the nonlocal generator inequality, the kernel mass, and transfer to the maximal operator. AI also proposed a kernel with $\alpha = 6/5$ and claimed mass bound

3.615749. **That proposed improvement is not verified in Lean.**

Code and proof; Palomar: squares.

Poisson and Heat: Harmonic Core, Original Tail

Write $\varphi(x) = c q(z)$, $z = |x|^2$, and $K(x) = c k(|x|^2)$. Use

$$k(z) = \begin{cases} q(a) + a^{n/2} q'(a)(s(z) - s(a)), & 0 < z < b, \\ q(z), & z \geq b, \end{cases}$$

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The inner branch touches tangentially at a , and rejoins at $b = \rho_n a$:

$$q(b) = q(a) + \frac{2a}{n} q'(a).$$

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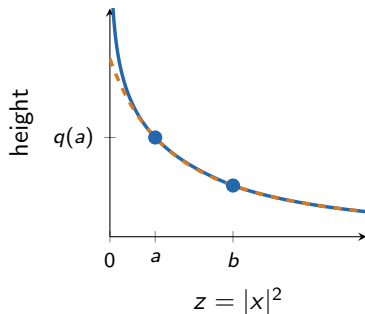
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	$q(z)$	c
Poisson	$(1+z)^{-(n+1)/2}$	$\Gamma((n+1)/2)/\pi^{(n+1)/2}$
Heat	e^{-z}	$\pi^{-n/2}$

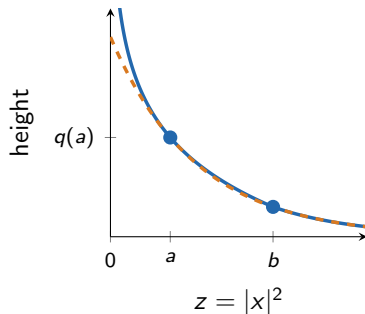
These Laplacian majorants attain the minimum mass. Details of their construction are omitted here.

Poisson and Heat Majorants in Dimension Two

Poisson

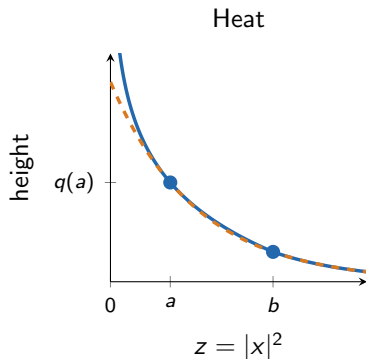
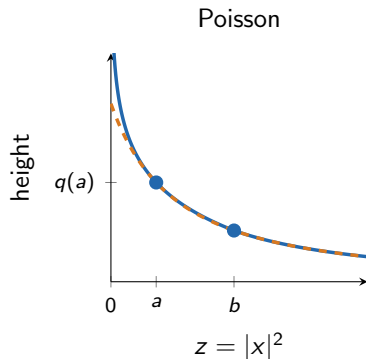


Heat



Blue: majorant. Dashed orange: original kernel. Heights are normalized by $q(a)$.

Poisson and Heat Majorants in Dimension Two



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$$\|P_*\|_{1 \rightarrow 1, \infty} \leq 1.02104 \dots, \quad \|H_*\|_{1 \rightarrow 1, \infty} \leq 1.094052 \dots$$

Touch at a , join at b , and retain the original nonzero tail.

The Rising Sun Lemma

For $f \geq 0$, define the one-sided maximal operator

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} f(y) dy.$$

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$$G(x) = \int_0^x (f(y) - \lambda) dy.$$

Then

$$E_\lambda = \{M^+ f > \lambda\} = \{x : G(x) < \sup_{y \geq x} G(y)\}.$$

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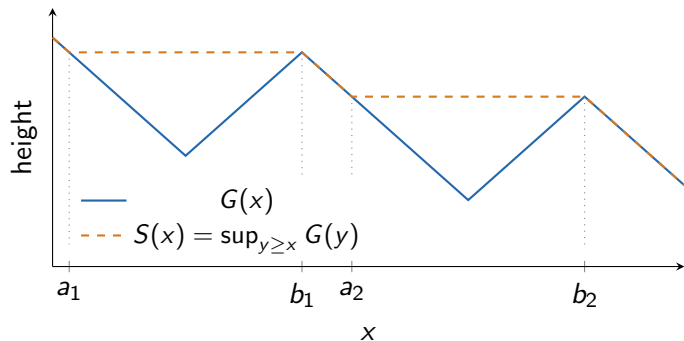
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The rising sun lemma decomposes this open set into disjoint intervals (a_j, b_j) with

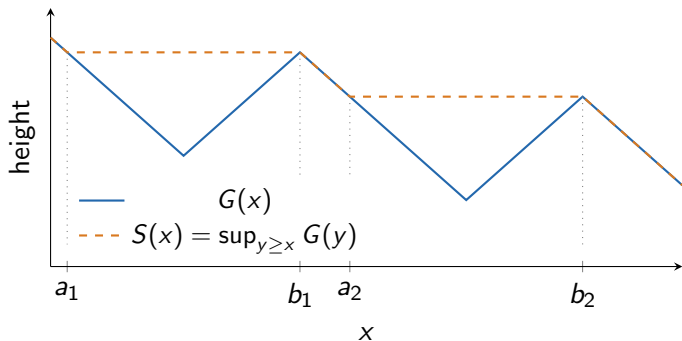
$$G(a_j) = G(b_j), \quad \int_{a_j}^{b_j} f = \lambda(b_j - a_j).$$

First take bounded compactly supported f , then approximate.

The Rising Sun Picture



The Rising Sun Picture



The gap $u = S - G$ is positive exactly on the shadow intervals.

Why Partial Balayage Is Natural

Rising sun gives

$$\lambda|E_\lambda| = \sum_j \int_{a_j}^{b_j} f \leq \|f\|_1.$$

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Set $u = S - G \geq 0$ and $\sigma = f + u'$. Then, a.e.,

$$f = \sigma - u', \quad 0 \leq \sigma \leq \lambda, \quad \sigma = \lambda \text{ on } \{u > 0\}.$$

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For x with $u(x) = 0$,

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Partial balayage replaces the one-dimensional derivative by other differential operators. The same controlled positivity set contains every superlevel set.

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Y. Lin. Two partial balayage principles; $c_2 < 4$.



Lean formalization; Palomar: two principles.
Lean: squares; Palomar: squares.