

A logarithmic refinement of CKN partial regularity

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Abstract

Let (u, p) be a suitable weak solution of the three-dimensional Navier–Stokes equations. We prove that its singular set has zero parabolic Hausdorff measure for the gauge $h(r) = r[\log(1/r)]^2$. The proof is generated by AI under my guidance.

1 Introduction

Consider the Navier–Stokes equations with viscosity normalized to one:

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0.$$

Caffarelli, Kohn, and Nirenberg [1] proved that the interior singular set of a suitable weak solution has zero one-dimensional parabolic Hausdorff measure. Logarithmic refinements of this conclusion have also been studied; see, for example, Lei and Ren [3]. This article explains an argument giving a squared logarithmic factor.

Let $\mathcal{D} \subset \mathbb{R}^3 \times \mathbb{R}$ be open. Denote by S the interior singular set: the points where u is not essentially bounded in any open spacetime neighborhood.

Write

$$C_r(x, t) = B_r(x) \times (t - r^2, t + r^2).$$

For an increasing gauge h , define the parabolic Hausdorff measure by

$$\mathcal{P}^h(A) = \liminf_{\delta \downarrow 0} \left\{ \sum_i h(r_i) : A \subset \bigcup_i C_{r_i}(z_i), \quad 0 < r_i < \delta \right\}.$$

Theorem 1. *Let (u, p) be a suitable weak solution on $\mathcal{D}$. Then*

$$\mathcal{P}^{r[\log(\frac{1}{r})]^2}(S) = 0.$$

The gauge is specified for sufficiently small r and extended increasingly to larger radii.

Main idea and physical interpretation. The local energy inequality compares the change in kinetic energy and the viscous dissipation with energy transported through the boundary of a cylinder. The pressure contributes to this transport. On a spatial scale r , the diffusion time is r^2 , which explains the parabolic cylinders used here. On each compact localization, the finite measure $|\nabla u|^2 dx dt$ records the viscous dissipation.

At a singular point, the scaled cubic velocity and pressure integral cannot become small. A local estimate says that this activity contracts at a smaller scale unless a certain mixed gradient integral pays for it. We use spatial exponent $12/7$ because Sobolev embedding then gives L^4 , and interpolation with L^2 matches the cubic velocity term. The resulting cost contains the square root

of the activity. The use of this mixed gradient exponent is consistent with earlier regularity criteria; see [4].

A one-scale covering argument does not compare the costs at different scales. Our trace estimate does: it sums the mixed dissipation against a measure supported on a hypothetically large singular set. The squared logarithmic factor corresponds to the weights $1/j$. Their sum diverges, while the trace estimate gives a finite integral of the same weighted cost. The logarithmic factor arises from this summation over scales, rather than from an additional physical assumption.

The proof has three steps.

1. Obtain a contraction estimate and show that persistent activity has infinite weighted dissipation cost.
2. Prove a trace estimate that bounds the total cost using the finite viscous dissipation.
3. Apply Frostman's lemma and compare cylinders with dyadic boxes to obtain a contradiction.

2 The cost of persistent activity

We use the standard notion of a suitable weak solution: the equations hold in distributions, locally

$$u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1, \quad p \in L_{x,t}^{3/2},$$

and the local energy inequality holds.

All estimates are local. After a fixed translation and rescaling, work on $B_4 \times (-4, 4) \Subset \mathcal{D}$, with centers in a compact set $L \subset B_1 \times (-1, 1)$. Choose all radii sufficiently small that the cylinders used below lie in $U = B_2 \times (-2, 2)$. A countable family of such localizations covers $\mathcal{D}$. Fixed rescaling preserves the nullity conclusion because $h(cr) \asymp h(r)$ for each fixed $c > 0$.

For $z = (x, t)$, define

$$\begin{aligned} G(r, z) &= r^{-2} \int_{C_r(z)} (|u|^3 + |p|^{3/2}) dy ds, \\ A(r, z) &= r^{-1} \operatorname{ess\,sup}_{|s-t| < r^2} \int_{B_r(x)} |u(y, s)|^2 dy, \\ E(r, z) &= r^{-1} \int_{C_r(z)} |\nabla u|^2 dy ds, \\ e(r, z) &= r^{-3/2} \int_{t-r^2}^{t+r^2} \left(\int_{B_r(x)} |\nabla u(y, s)|^{12/7} dy \right)^{7/6} ds. \end{aligned}$$

These quantities are invariant under Navier–Stokes scaling. Spatial Hölder gives $e(r, z) \leq CE(r, z)$. A standard cutoff in the local energy inequality gives

$$A(r/2, z) + E(r/2, z) \leq C(G(r, z)^{2/3} + G(r, z)).$$

Here the quadratic cutoff term gives $G^{2/3}$, and the cubic and pressure flux terms give G ; for the latter use Hölder with the pressure exponent $3/2$. See also the local energy estimates in [2].

The standard velocity–pressure epsilon-regularity criterion implies that, for a fixed $\varepsilon_* > 0$,

$$G(r, z) \geq \varepsilon_* \quad \text{at every } z \in S \cap L \text{ and every small } r.$$

To apply the backward-cylinder formulation to our symmetric cylinders, use the backward cylinder of radius r with top time $t + r^2/8$. It lies in $C_r(z)$, and its smaller regularity cylinder contains an open neighborhood of z . Thus smallness of $G(r, z)$ implies regularity at z . The precise criterion is recorded in [2, Theorem 1.1].

Lemma 2 (Scale recurrence). *There are fixed constants K , $\theta = 2^{-M} < 1/(2K)$, and $C_* > 0$ such that*

$$G(\theta r, z) \leq \frac{1}{4}G(r, z) + C_* \sqrt{G(r, z)} e(r/K, z) \quad \text{whenever } G(r, z) \geq \varepsilon_*.$$

Proof. We first show that for $0 < s \leq \rho/2$, suppressing the center,

$$G(s) \leq C \frac{s}{\rho} G(\rho) + C \left(\frac{\rho}{s} \right)^2 A(\rho)^{1/2} e(\rho).$$

At each time, let b be the spatial mean of u on B_ρ . Sobolev–Poincaré and interpolation give

$$\|u - b\|_4 \leq C \|\nabla u\|_{12/7}, \quad \|u - b\|_3^3 \leq \|u - b\|_2 \|u - b\|_4^2.$$

All norms here are over B_ρ . There is no radius factor in the first inequality because $1/(12/7) - 1/4 = 1/3$. Taking the L^2 norm uniformly in time and integrating the other factor gives

$$\int_{C_\rho} |u - b|^3 \leq C \left(\operatorname{ess\,sup}_s \int_{B_\rho} |u|^2 \right)^{1/2} \int_{t-\rho^2}^{t+\rho^2} \left(\int_{B_\rho} |\nabla u|^{12/7} \right)^{7/6} ds = C \rho^2 \sqrt{A(\rho)} e(\rho).$$

Jensen’s inequality bounds the mean contribution on C_s , after division by s^2 , by $C(s/\rho)G(\rho)$.

For the pressure, decompose at each time

$$p_1 = R_i R_j [\mathbf{1}_{B_\rho}(u_i - b_i)(u_j - b_j)], \quad p_2 = p - p_1,$$

where R_i are the Riesz transforms and repeated indices are summed. The pressure equation and incompressibility imply that p_2 is harmonic on B_ρ . The standard estimates give

$$\int_{\mathbb{R}^3} |p_1|^{3/2} \leq C \int_{B_\rho} |u - b|^3, \quad \int_{B_s} |p_2|^{3/2} \leq C(s/\rho)^3 \int_{B_\rho} |p_2|^{3/2}.$$

Integrating over the smaller time interval and using the preceding velocity bound proves the claimed estimate for $G(s)$.

Now set $\rho = r/K$ and $s = \theta r$. Since

$$G(r/K) \leq K^2 G(r), \quad A(r/K) \leq (K/2) A(r/2),$$

the local energy bound gives

$$G(\theta r) \leq CK^3 \theta G(r) + C_{K,\theta} (G(r)^{1/3} + G(r)^{1/2}) e(r/K).$$

Choose a dyadic θ so that $CK^3 \theta \leq 1/4$. For $G(r) \geq \varepsilon_*$, the factor $G(r)^{1/3}$ is bounded by a constant times $G(r)^{1/2}$. This proves the recurrence. $\square$

Lemma 3 (Persistent activity). *Put $r_j = R\theta^j$, where R is small, and let $G_j(z) = G(r_j, z)$, $e_j(z) = e(r_j/K, z)$. If $G_j(z) \geq \varepsilon_*$ for every j , then*

$$\sum_{j \geq 2} \frac{e_j(z)}{j \sqrt{G_j(z)}} = \infty.$$

Proof. Lemma 2 and $\log(1+s) \leq s$ give

$$\log(G_{j+1}/\varepsilon_*) - \log(G_j/\varepsilon_*) \leq -\log 4 + 4C_* \frac{e_j}{\sqrt{G_j}}.$$

Multiply by $1/j$ and sum. Since $G_j \geq \varepsilon_*$, summation by parts gives

$$\begin{aligned} \sum_{j=2}^N \frac{\log(G_{j+1}/\varepsilon_*) - \log(G_j/\varepsilon_*)}{j} &= \frac{\log(G_{N+1}/\varepsilon_*)}{N} - \frac{1}{2} \log(G_2/\varepsilon_*) \\ &\quad + \sum_{j=3}^N \frac{\log(G_j/\varepsilon_*)}{j(j-1)} \\ &\geq -\frac{1}{2} \log(G_2/\varepsilon_*). \end{aligned}$$

Therefore

$$4C_* \sum_{j=2}^N \frac{e_j}{j\sqrt{G_j}} \geq (\log 4) \sum_{j=2}^N \frac{1}{j} - \frac{1}{2} \log(G_2/\varepsilon_*) \rightarrow \infty.$$

□

3 A trace estimate for the dissipation

We now prove the estimate that sums the dissipation cost over all scales. Define the finite measure

$$\nu(A) = \int_{A \cap U} |\nabla u|^2 dx dt.$$

In the following formulas, $|\nabla u|$ is extended by zero outside U ; no extension of u is needed. Fix a nested parabolic dyadic grid. A level- n box $Q = X \times I$ has spatial side $\ell_n = 2^{-n}$ and temporal length ℓ_n^2 . Use half-open boxes so that each level partitions spacetime. Define

$$E_Q = \frac{\nu(Q)}{\ell_n}, \quad e_Q = \ell_n^{-3/2} \int_I \left(\int_X |\nabla u|^{12/7} dx \right)^{7/6} dt.$$

These are the box versions of E and e . A quotient with $\nu(Q) = 0$ is set to zero; in that case $e_Q = 0$.

Lemma 4 (Dissipation trace). *Let μ be a finite nonnegative measure. Suppose that $F(n) > 0$ is nondecreasing for $n \geq n_0$ and*

$$\mu(Q) \leq C_\mu \ell_n F(n) \quad \text{for every level-}n \text{ box, } n \geq n_0.$$

Then

$$\sum_{n \geq n_0} \sum_{\text{level}(Q)=n} \mu(Q) \frac{e_Q}{\sqrt{E_Q F(n)}} \leq C \sqrt{\mu(\mathbb{R}^4) \nu(\mathbb{R}^4)}.$$

The constant depends only on C_μ and the spatial dimension.

Proof. The proof has two steps: a local Carleson estimate, followed by a summation over the ratio of the two masses.

Step 1: an estimate inside each root. Put

$$\beta_Q = \sqrt{\frac{\mu(Q)\ell_n}{F(n)}} e_Q.$$

At a fixed time t , let $I_n(t)$ be its level- n grid interval. For a level- n spatial cube X , define

$$\alpha_X(t) = \frac{\ell_n^{5/2}}{\sqrt{F(n)}} \sqrt{\mu(X \times I_n(t))}.$$

For a spatial ancestor X_0 at level $k \geq n_0$, Cauchy–Schwarz over the $(\ell_k/\ell_n)^3$ spatial descendants and nesting of the time intervals give

$$\begin{aligned} \sum_{X \subseteq X_0} \alpha_X(t) &\leq \ell_k^{3/2} \sqrt{\mu(X_0 \times I_k(t))} \sum_{n \geq k} \frac{\ell_n}{\sqrt{F(n)}} \\ &\leq 2 \frac{\ell_k^{5/2}}{\sqrt{F(k)}} \sqrt{\mu(X_0 \times I_k(t))} \leq C|X_0|. \end{aligned}$$

Thus the coefficients form a spatial Carleson sequence, uniformly in t . Apply the standard dyadic Carleson embedding at exponent $7/6$ to $|\nabla u(\cdot, t)|^{12/7}$, restricted to X_0 . Writing the spatial averages explicitly, it gives

$$\sum_{X \subseteq X_0} \alpha_X(t) \left(\frac{1}{|X|} \int_X |\nabla u(x, t)|^{12/7} dx \right)^{7/6} \leq C \int_{X_0} |\nabla u(x, t)|^2 dx.$$

Since $e_Q = \ell_n^2 \int_I \left(\frac{1}{|X|} \int_X |\nabla u(x, t)|^{12/7} dx \right)^{7/6} dt$, integrating over the time interval of a root $Q_0 = X_0 \times I_0$ and using Tonelli proves

$$\sum_{Q \subseteq Q_0} \beta_Q \leq C\nu(Q_0).$$

The sum includes Q_0 and its descendants. The essential point is that this estimate holds inside every root.

Step 2: summation by the mass ratio. Fix $\lambda > 0$ and consider boxes with $\nu(Q) > 0$ and $\mu(Q) > \lambda\nu(Q)$. Their maximal ancestors satisfying the same condition, within the level- n_0 roots, are disjoint. Applying Step 1 on these ancestors gives

$$\sum_{\substack{\nu(Q) > 0 \\ \mu(Q) > \lambda\nu(Q)}} \beta_Q \leq C \min \left\{ \nu(\mathbb{R}^4), \frac{\mu(\mathbb{R}^4)}{\lambda} \right\}.$$

Layer cake therefore yields

$$\begin{aligned} \sum_{\nu(Q) > 0} \beta_Q \sqrt{\frac{\mu(Q)}{\nu(Q)}} &= \frac{1}{2} \int_0^\infty \lambda^{-1/2} \sum_{\substack{\nu(Q) > 0 \\ \mu(Q) > \lambda\nu(Q)}} \beta_Q d\lambda \\ &\leq 2C \sqrt{\mu(\mathbb{R}^4)\nu(\mathbb{R}^4)}. \end{aligned}$$

The assertion is immediate if either total mass is zero. Finally,

$$\beta_Q \sqrt{\frac{\mu(Q)}{\nu(Q)}} = \mu(Q) \frac{e_Q}{\sqrt{E_Q F(n)}},$$

which proves the lemma. □

4 Proof of Theorem 1

Fix the compact localization L from Section 2. Choose a small dyadic R and use $r_j = R\theta^j$ as above. We argue by contradiction, assuming $\mathcal{P}^h(S \cap L) > 0$.

Step 1: a measure on the persistent set. For each fixed r , the function $G(r, z)$ is continuous in z , as an integral of a locally integrable density over translated cylinders. Thus

$$T = \{z \in L : G(r_j, z) \geq \varepsilon_* \text{ for every } j \geq 0\}$$

is compact and contains $S \cap L$. Frostman's lemma for a doubling gauge in the parabolic metric supplies a probability measure supported on T such that

$$\mu(C_r(z)) \leq Ch(r) \quad \text{for all centers and all sufficiently small } r.$$

See [5] for the gauge formulation of Frostman's lemma.

Step 2: comparison of cylinders and boxes. Use the standard one-third trick: three adjacent nested grids in each spatial coordinate and three in time give 81 parabolic product grids. For a dyadic ρ , every $C_\rho(z)$ is contained in a box with spatial side 16ρ and temporal length $(16\rho)^2$ in one of these grids. This is the usual adjacent-grid containment property; see [6] for the general construction. Put $\rho = r_j/K$ and choose such a box Q . The box has level $n_j = Mj + c$, where c is a fixed integer. Moreover $Q \subset C_{r_j/2}(z)$, since

$$16\sqrt{3}\rho < r_j/2, \quad 256\rho^2 < (r_j/2)^2.$$

Because Q contains the open cylinder centered at z , it is that grid's unique level- n_j box containing z . Monotonicity of the unnormalized integrals and the local energy bound now give, for $z \in T$,

$$e_j(z) \leq 16^{3/2}e_Q, \quad E_Q \leq 16E(r_j/2, z) \leq C(G_j(z)^{2/3} + G_j(z)) \leq C_*G_j(z).$$

Consequently

$$\frac{e_j(z)}{\sqrt{G_j(z)}} \leq C \frac{e_Q}{\sqrt{E_Q}},$$

with the zero convention already specified.

Step 3: finite trace versus infinite cost. Each fine box lies in a cylinder of comparable radius. The Frostman bound therefore implies, in every one of the grids,

$$\mu(Q) \leq C\ell_n F(n), \quad F(n) = n^2,$$

because $h(C\ell_n) \asymp \ell_n n^2$. Apply Lemma 4 and write $Q_n(z)$ for the box containing z . It follows that

$$\int_T \sum_{n \geq n_0} \frac{e_{Q_n(z)}}{n\sqrt{E_{Q_n(z)}}} d\mu(z) < \infty.$$

Since $n_j = Mj + c$, its weight is comparable to $1/j$. The comparison in Step 2 and summation over all 81 grids thus give

$$\int_T \sum_{j \geq j_0} \frac{e_j(z)}{j\sqrt{G_j(z)}} d\mu(z) < \infty.$$

Using the sum over the grids avoids having to choose a grid measurably for each (z, j) . But Lemma 3 makes the integrand infinite at every point of T , and $\mu(T) = 1$. This contradiction proves $\mathcal{P}^h(S \cap L) = 0$. A countable localization completes the proof. $\square$

5 A short extension

The preceding proof uses only the harmonic weights $1/j$. Inserting additional logarithmic factors in the gauge leads to slower divergent weights and gives the following extension.

Proposition 5. *Let $L_1(r) = \log(1/r)$ and $L_{m+1}(r) = \log L_m(r)$. For every fixed integer $k \geq 2$,*

$$\mathcal{P}^{h_k}(S) = 0, \quad h_k(r) = r \prod_{m=1}^k L_m(r)^2,$$

Proof. Repeat the proof with

$$F(n) = n^2 \prod_{m=1}^{k-1} (\log_m n)^2.$$

Here $\log_1 n = \log n$ and $\log_{m+1} n = \log(\log_m n)$. For large n , F increases and $F(Mj + c) \asymp F(j)$. The reciprocals $1/(j \prod_{m=1}^{k-1} \log_m j)$ decrease and have divergent sum by the integral test. Repeating the summation by parts in Lemma 3 with these explicit reciprocals, and applying Lemma 4, gives the same contradiction. $\square$

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