NEW PROGRESS ON BESICOVITCH'S 1/2 PROBLEM

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Abstract. Let H be a real Hilbert space. We prove that if $a > 0.6934$ and $E \subset H$ is Borel, $\mathcal{H}^1(E) < \infty$, and $\Theta^1_*(E, x) \geq a$ for $\mathcal{H}^1$-almost every $x \in E$, then E is countably 1-rectifiable. Most of the proof and its formalization were generated by AI under human direction.

1. Introduction and notation

Definition 1.1 (Straight measure)**.** A Borel measure μ on H is *straight*[1] if $\mu(A) \leq \operatorname{diam} A$ for every Borel set $A \subset H$.

Definition 1.2 (Rectifiability threshold)**.** Define

$$\sigma_1(H) := \inf\big\{a \geq 0 : \Theta^1_*(E, x) \geq a \text{ for } \mathcal{H}^1\text{-a.e. } x \in E \implies E \text{ is countably 1-rectifiable}\big\},$$

where E ranges over Borel sets with $\mathcal{H}^1(E) < \infty$.

Definition 1.3 (Besicovitch pair condition)**.** The space H satisfies BPC(β) if, for every straight measure μ, there is $\tau > 0$ such that for every $\lambda > 0$ there is $\delta > 0$ with the following property. If E_1, E_2 are nonempty Borel sets with $0 < \operatorname{dist}(E_1, E_2) < \delta$ and $\mu(B(x, r)) > 2\beta r$ for $x \in E_1 \cup E_2$ and $0 < r < \lambda$, then there is an open set V meeting both E_1 and E_2 such that $\mu(V \setminus (E_1 \cup E_2)) > \tau \operatorname{diam} V$. We say that V *leaks*[1] from $E_1 \cup E_2$.

Preiss–Tišer [PT92] showed that BPC(β) $\implies \sigma_1(H) \leq \beta$. Thus we seek to establish BPC(β) for β as close to $1/2$ as possible. De Lellis, Glaudo, Massaccesi, and Vittone [De +26] introduced a optimization framework and proved $\sigma_1(\mathbb{R}^2) \leq 0.7$. We propose a different optimization problem to obtain $\sigma_1(H) \leq 0.6934$.

2. The six-point game

Definition 2.1 (Admissible configuration and score)**.** An admissible six-point configuration at $s \in (1/2, 1]$ consists of blue points o, p_1, p_2 and red points e, q_1, q_2 such that $d(o, e) = 1$, $d(o, p_i), d(e, q_i) \leq 1$, and $d(p_1, p_2), d(q_1, q_2) \geq 2s$. The six labelled points need not be distinct. A move selects at least one point of each colour and assigns radii $r_i \in [0, 1]$ to each selected point such that $r_i + r_j \leq d(x_i, x_j)$ whenever $i \neq j$ and x_i, x_j have the same

[1]The words *straight* and *leak* are not standard. They were suggested by AI and retained by the author because they are intuitive.

colour, which guarantees that balls of the same colour are disjoint. The virtual diameter and score of a move are

$$D := \max_{i,j}\big(d(x_i, x_j) + r_i + r_j\big), \qquad \mathsf{S}_s := \sum_i r_i - \frac{D}{2s}.$$

The six-point property holds at s if every admissible configuration has a move with $\mathsf{S}_s \geq 0$.

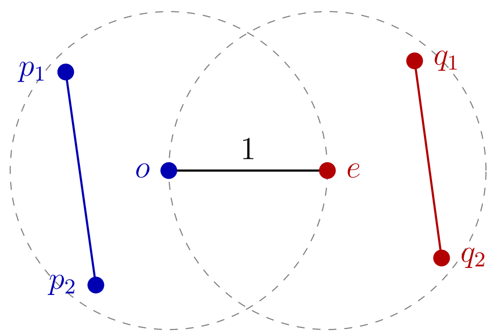

FIGURE 1. The six-point game. Each sibling segment has length at least $2s$.

Theorem 2.1. *If the six-point property holds at s, then H satisfies* BPC(β) *for every $\beta > s$. Consequently, $\sigma_1(H) \leq s$.*

Proof. Fix $\beta > s$ and choose $q \in (s/\beta, 1)$. Choose $\tau > 0$ so small that

$$\tau(2 + q^{-1}) < 2(\beta - s/q), \qquad (1 + \tau)s < \beta q.$$

Let μ be straight. Given $\lambda > 0$, set $\delta = \lambda$ and take E_1, E_2 as in Definition 1.3. Write $d = \operatorname{dist}(E_1, E_2)$. Choose roots $x_i \in E_i$ with $d \leq \ell := d(x_1, x_2) < d/q$ and put $U = B(x_1, d) \cup B(x_2, d)$. If U leaks, then we are done. Otherwise,

$$\mu(U \setminus (E_1 \cup E_2)) \leq \tau \operatorname{diam} U < \tau(2 + q^{-1})d.$$

Each root ball misses the opposite set. Since $d < \lambda$, the density hypothesis gives,

$$\begin{aligned} 2\beta d &< \mu(B(x_i, d)) \\ &= \mu(B(x_i, d) \cap E_i) + \mu(B(x_i, d) \setminus E_i) \\ &\leq \mu(B(x_i, d) \cap E_i) + \mu(U \setminus (E_1 \cup E_2)) \\ &< \mu(B(x_i, d) \cap E_i) + \tau(2 + q^{-1})d. \end{aligned}$$

Thus straightness implies

$$\operatorname{diam}(B(x_i, d) \cap E_i) \geq \mu(B(x_i, d) \cap E_i) > 2\beta d - \tau(2 + q^{-1})d > 2sd/q > 2s\ell.$$

Hence we can choose $x_{i1}, x_{i2} \in B(x_i, d) \cap E_i$ with $d(x_{i1}, x_{i2}) > 2s\ell$. Dividing all distances by ℓ gives an admissible six-point configuration, with the points in E_1 blue and those in E_2 red.

Take a nonnegative-score move. Denote its selected centres by y_j, its radii by r_j, and its virtual diameter by D. The six-point property gives $D \leq 2s \sum_j r_j$. Since the move selects both colours, we have $D > 0$ and hence $\sum_j r_j > 0$. For each $r_j > 0$, consider the ball $B(y_j, q\ell r_j)$. This ball misses the opposite set and since $q\ell r_j < d < \lambda$, $\mu(B(y_j, q\ell r_j)) > 2\beta q\ell r_j$. Balls of the same colour are disjoint, because $q\ell(r_i + r_j) \le q\, d(y_i, y_j) < d(y_i, y_j)$. Consequently, a point lying in two such balls does not belong $E_1 \cup E_2$.

It is possible that the radius of a selected centre is zero in this move, so for all $\varepsilon > 0$, we consider the set $V_\varepsilon = \bigcup_j B(y_j, q\ell r_j + \varepsilon)$. This open set meets both E_1 and E_2, and $\operatorname{diam} V_\varepsilon \le \ell D + 2\varepsilon$. The aforementioned overlap observation gives the pointwise bound $\sum_{j:r_j>0} \mathbf{1}_{B(y_j,q\ell r_j)} \le \mathbf{1}_{V_\varepsilon} + \mathbf{1}_{V_\varepsilon \setminus (E_1 \cup E_2)}$. Integrating with respect to μ and assuming that V_ε does not leak for any $\varepsilon > 0$, we obtain

$$\begin{aligned} 2\beta q\ell \sum_j r_j &< \sum_{j:r_j>0} \mu(B(y_j, q\ell r_j)) \\ &\le \mu(V_\varepsilon) + \mu(V_\varepsilon \setminus (E_1 \cup E_2)) \\ &\le (1+\tau) \operatorname{diam} V_\varepsilon \\ &\le (1+\tau)(\ell D + 2\varepsilon). \end{aligned}$$

Letting $\varepsilon \downarrow 0$ and using $D \le 2s \sum_j r_j$, we obtain $2\beta q\ell \sum_j r_j \le 2(1+\tau)s\ell \sum_j r_j$. As $0 < \sum_j r_j$, this inequality contradicts with $(1+\tau)s < \beta q$. Hence, one of the V_ε leaks and BPC(β) holds. □

3. AI-generated formalization in Lean

Theorem 3.1 (Machine-checked bound)**.** $\sigma_1(H) \le 0.6934$.

The author does not understand the proof of this theorem. However, the proof has been verified in Lean and uses Theorem 2.1. The code is available at https://github.com/CoolRmal/Besicovitchs-1-2. The repository includes a Lean Comparator check. The result is also registered on Palomar at https://palomar-registry.org/entry?id=PALOMAR-2026-09-02-000011&version=1.

Theorem 3.1 is also compelling when viewed in comparison with the method of Preiss–Tišer [PT92]: their argument uses information from only one of the sets E_1 and E_2, whereas Theorem 2.1 exploits information from both.

Let s_* denote the infimum of all s for which the six-point property holds at s. An initial AI-generated proof established that s_* is algebraic and $s_* \approx 0.6933064218$. Formalizing that argument requires substantially more computation, so the author chose the weaker threshold 0.6934.

4. Potential improvements

In the proof of theorem 2.1, we consider two cases whether $B(x_1, d) \cup B(x_2, d)$ leaks, and we extract six points in the case that it does not leak. We can then also consider two cases whether $\bigcup_{i=1}^{6} B(x_i, d)$ leaks. In the case that this union of six balls does not leak, for each colour, we can extract one root, two children, and four grandchildren, suggesting a fourteen-point game. An AI-guided numerical search suggested that this larger game may remain nonnegative near $s = 0.65$, but this is not verified in Lean as it is much more computationally heavy.

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References

[De +26] Camillo De Lellis, Federico Glaudo, Annalisa Massaccesi, and Davide Vittone. "Besicovitch's 1/2 Problem and Linear Programming". In: *Transactions of the American Mathematical Society* 379.7 (2026), pp. 5051–5098. DOI: 10.1090/tran/9609. arXiv: 2404.17536.

[PT92] David Preiss and Jaroslav Tišer. "On Besicovitch's 1/2-Problem". In: *Journal of the London Mathematical Society*. 2nd ser. 45.2 (1992), pp. 279–287. DOI: 10.1112/jlms/s2-45.2.279.